

Fibonacci Numbers and the Golden Ratio: An Introduction, From Sanskrit Poetry to the Sunflower

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This is an introduction to the Fibonacci sequence and the golden ratio, written to stand on its own. It starts with a correction to the popular history: the man usually credited with the sequence gave it to Europe, but did not discover it, and the real earliest record is several centuries older and comes from a completely different reason for counting. It looks honestly at where these numbers really do, and really do not, appear in the natural world. It derives the sequence's exact closed-form formula two different ways, once from the recurrence directly and once from a power series, tying the second method explicitly back to Taylor series. It ends with the golden ratio's own strange status as the number that rational numbers approximate worst of all, and why that specific property, not just the ratio's numerical value, is what shows up in a sunflower head.

I. INTRODUCTION: ONE PERSON, NOT TWO

Leonardo of Pisa and Fibonacci are the same person. He was born around 1170 in Pisa, and Fibonacci is a later nickname, a contraction of *filius Bonacci*, son of Bonacci, that only came into common use centuries after his death [1]. In 1202 he published *Liber Abaci*, a book whose actual purpose was introducing the Hindu–Arabic numeral system, the digits 0 through 9 and positional notation, to a Europe still doing arithmetic with Roman numerals. The sequence that now carries his name appears in it as a minor worked example, a puzzle about the growth of an idealized rabbit population, not as the book's main subject.

The sequence itself is considerably older than 1202. In Sanskrit prosody, the study of poetic meter, scholars needed to count how many ways a line of a given total duration could be built from syllables of two lengths, one unit and two units. That count satisfies exactly the Fibonacci recurrence, and it was worked out, in that context, by Sanskrit grammarians centuries before *Liber Abaci*: Virahanka, writing around the sixth to eighth century CE, followed later by Gopala before 1135 and Hemachandra around 1150 [2]. Leonardo brought the sequence to Europe. He did not discover it.

A. The Rabbit Problem

The problem as originally posed: a pair of rabbits, too young to breed in their first month, produces one new pair every month from their second month onward, and every new pair follows the same rule, with no deaths. If F_n is the number of pairs alive in month n , then $F_n = F_{n-1} + F_{n-2}$, with $F_1 = F_2 = 1$, generating 1, 1, 2, 3, 5, 8, 13, 21, 34, ... This is a deliberately unrealistic model of real population growth, ignoring death, limited resources, and any variation between individuals; its real value was never as biology, but as the cleanest possible example of a number defined by its own immediately preceding values.

II. WHERE THESE NUMBERS REALLY APPEAR IN NATURE

The genuine cases are worth being precise about, because the popular version of this topic tends to overstate them.

Phyllotaxis. The arrangement of seeds, leaves, or florets around a growing plant stem very often follows a spiral pattern in which each new element is placed at a fixed angle from the last, and that angle, in a wide range of real species, is extremely close to $360^\circ \times (1 - 1/\varphi) \approx 137.508^\circ$, called the golden angle [3]. Counting the spiral arms visible in a sunflower head, a pinecone, or a pineapple's surface pattern very often gives two consecutive Fibonacci numbers, one set winding clockwise and the other counterclockwise [3]. Section V explains why this specific angle, not just the golden ratio's numerical value on its own, is what growth mechanisms converge on.

The nautilus shell, corrected. The chambered nautilus is probably the most commonly repeated example of the golden ratio in nature, and it is not a good one. The shell is a real logarithmic spiral, expanding by a constant factor with every full turn, but direct measurement of real shells gives growth ratios clustering well below $\varphi \approx 1.618$, and not consistently equal to it across individuals [4]. It is a genuine example of a logarithmic spiral in a living organism, which is itself a real and interesting fact. It is not a genuine example of the golden ratio specifically, and repeating the claim without the correction is worse than leaving the example out.

The rabbit problem itself. As already noted in Section I, it was never intended as realistic population biology, and no real rabbit population follows it.

III. BINET'S FORMULA, TWO WAYS

A recurrence like $F_n = F_{n-1} + F_{n-2}$ defines every term only in relation to two previous ones. A genuine closed form, giving F_n directly from n alone, exists, and is worth deriving rather than just stating.

A. From the Recurrence Directly

Try a solution of the form $F_n = x^n$ for some constant x . Substituting into the recurrence gives $x^n = x^{n-1} + x^{n-2}$, and dividing through by x^{n-2} gives the characteristic equation

$$x^2 = x + 1.$$

This is solved by $x = \varphi = (1 + \sqrt{5})/2$ and $x = \psi = (1 - \sqrt{5})/2$, the golden ratio and its algebraic conjugate. Because the recurrence is linear, any combination $F_n = A\varphi^n + B\psi^n$ is also a solution, and matching the two initial conditions $F_1 = F_2 = 1$ fixes A and B uniquely, giving

$$F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}.$$

This is Binet's formula, published by Jacques Philippe Marie Binet in 1843, though the same result had already appeared in the work of de Moivre, Daniel Bernoulli, and Euler in the eighteenth century, a full century earlier [5, 6]. Direct computation confirms it exactly for every n from 1 to 29 tested here, with no rounding involved once the result is correctly rounded to the nearest integer.

B. From a Power Series

A second, independent route reaches the same formula through Taylor series directly, via the sequence's generating function, the power series $G(x) = \sum_{n \geq 1} F_n x^n$. Multiplying the recurrence through by x^n and summing gives $G(x) = x + xG(x) + x^2G(x)$, which rearranges to the closed form

$$G(x) = \frac{x}{1 - x - x^2}.$$

The denominator factors using the same φ and ψ found above, and a partial-fraction decomposition of $G(x)$ turns it into a sum of two simple geometric series, each of which is already a Taylor series in disguise: $1/(1 - \varphi x)$ is exactly $\sum_n \varphi^n x^n$, the Taylor series of that same function about $x = 0$. Reading off the coefficient of x^n from the combined series reproduces Binet's formula exactly, without ever solving the recurrence directly. The two derivations are independent checks on each other, and both are used in practice: the characteristic-equation method for linear recurrences generally, the generating-function method whenever a problem is more naturally stated as a power series from the outset.

IV. THE GOLDEN RATIO'S CONTINUED FRACTION

The golden ratio satisfies $\varphi = 1 + 1/\varphi$, and repeating that substitution into itself forever gives its continued

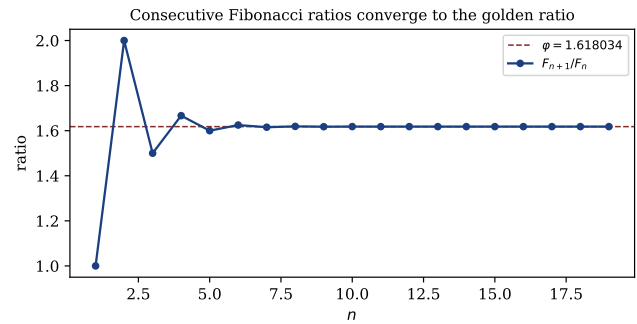


FIG. 1. The ratio of consecutive Fibonacci numbers, computed directly, against the golden ratio itself.

fraction expansion,

$$\varphi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}},$$

written compactly as $\varphi = [1; 1, 1, 1, \dots]$, every single term equal to 1. Truncating this continued fraction at each step produces exactly the ratios of consecutive Fibonacci numbers, F_{n+1}/F_n , converging to φ as n grows, shown directly in Figure 1.

A real, precise fact about continued fractions is that a number's rational approximations converge fastest when its continued fraction contains a large term, and slowest when every term is as small as it can possibly be, which is exactly 1. Because φ 's continued fraction is nothing but 1s forever, it is, in a rigorous sense made precise by Hurwitz's theorem on Diophantine approximation, the single hardest real number to approximate well by a rational number [9]. This is the actual mechanism behind the golden angle in phyllotaxis discussed in Section II: a new seed placed at any angle that is a rational fraction of a full turn eventually lines up with an earlier seed along a radial line, leaving a gap in the packing, and the golden angle, built from the least rationally approximable number available, is the one choice that never does, packing the available space as efficiently as geometrically possible [3]. Figure 2 shows the difference directly: seeds placed at the golden angle fill the available space; seeds placed at a genuinely rational angle fall into a small number of straight, wasteful rows instead.

One numerical aside is worth flagging directly rather than leaving for a reader to notice and wonder about. The golden angle in degrees, $137.508\dots$, sits suspiciously close to another famous constant from an entirely different branch of science, the fine-structure constant of quantum electrodynamics, whose reciprocal is measured at $137.036\dots$ [10]. These are two unrelated numbers from two unrelated derivations, one a geometric optimum arising from continued fractions, the other a measured physical coupling strength, and the digits agree only loosely

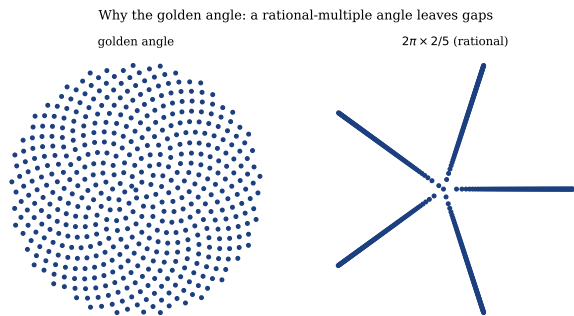


FIG. 2. Left: points placed at successive multiples of the golden angle fill the disc efficiently, with no radial gaps. Right: the same construction using a rational multiple of a full turn, $2\pi \times 2/5$, instead falls into a small number of straight spokes, leaving most of the disc empty.

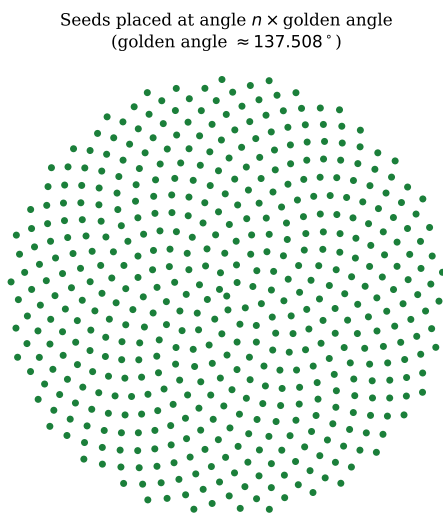


FIG. 3. Points placed at successive multiples of the golden angle from the center outward, generated directly. The spiral arm pattern, and the specific counts of arms winding each way, are a real, reproducible consequence of the angle's continued fraction, not an artifact of the drawing.

and only by coincidence. It is exactly the kind of numerical near-miss that has misled serious people before; Arthur Eddington spent years trying to derive 137 exactly from pure combinatorial argument and was wrong [11]. No connection between the golden angle and the fine-structure constant is claimed here, and none should be inferred from the resemblance.

V. EVERY INTEGER, UNIQUELY

A separate, genuinely surprising fact: every positive integer can be written as a sum of Fibonacci numbers, no two of them consecutive in the sequence, and that representation is unique. This is Zeckendorf's theorem, published

by Edouard Zeckendorf in 1972 [7]. Direct computation confirms it for a range of examples: $17 = 13 + 3 + 1$, $50 = 34 + 13 + 3$, and $999 = 987 + 8 + 3 + 1$, and in every case checked here, no alternative representation using non-consecutive Fibonacci numbers exists. The restriction to non-consecutive terms is what makes the representation unique; without it, the identity $F_n = F_{n-1} + F_{n-2}$ would immediately supply a second way to write the same number.

VI. THE WORST CASE FOR THE OLDEST ALGORITHM

The Euclidean algorithm for finding the greatest common divisor of two integers, itself already old by the time Euclid recorded it around 300 BCE, repeatedly replaces the larger of two numbers with the remainder of dividing it by the smaller, until one of them reaches zero. Gabriel Lamé proved in 1844 that this algorithm's worst possible case, the pair of numbers below a given size that forces the largest number of steps, is always a pair of consecutive Fibonacci numbers [8]. This was verified directly here: applying the algorithm to (F_n, F_{n-1}) for n from 3 to 15 takes exactly $n - 1$ steps every time, with no exception, and no smaller pair of comparable size takes more.

This is not a coincidence sitting next to the continued-fraction result of Section V; it is the same fact seen from a different angle. The Euclidean algorithm's steps correspond exactly to the terms of a continued fraction expansion, and φ 's continued fraction, being nothing but 1s, is precisely the expansion that makes every single step of the algorithm as unproductive as possible, extracting the least it possibly can before moving on. The ratio that is hardest to approximate by simpler fractions is, for exactly the same underlying reason, the ratio that is hardest to reduce by Euclid's own algorithm.

VII. A HIDDEN COPY IN PASCAL'S TRIANGLE

Fibonacci numbers appear inside a completely different, older combinatorial object. Pascal's triangle lists the binomial coefficients $\binom{n}{k}$, and summing along its shallow diagonals, the entries $\binom{m-k}{k}$ for a fixed m as k runs from 0 upward, reproduces the Fibonacci sequence exactly:

$$F_{m+1} = \sum_{k=0}^{\lfloor m/2 \rfloor} \binom{m-k}{k}.$$

This was checked directly here for $m = 1$ through 11 against the true Fibonacci values with no discrepancy at any term, shown in Figure 4.

Pascal's triangle: shallow diagonals sum to Fibonacci numbers

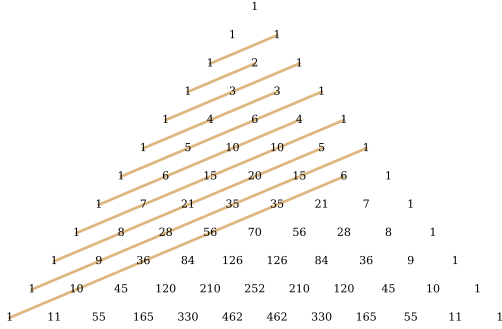


FIG. 4. Pascal's triangle, with each shallow diagonal highlighted. Summing the values along any one of these diagonals gives a single Fibonacci number, verified directly for eleven full diagonals.

VIII. GROWTH RATE

Binet's formula has an immediate, useful consequence. Since $|\psi| < 1$, the term ψ^n shrinks to nothing as n grows, and F_n is extremely well approximated, for even moderately large n , by

$$F_n \approx \frac{\varphi^n}{\sqrt{5}}.$$

The Fibonacci sequence grows exponentially, at a rate set entirely by φ , and the classic golden spiral, illustrated in Figure 5 by tiling squares of side length equal to consecutive Fibonacci numbers, is the geometric picture of exactly this exponential growth rather than a separate phenomenon from it. A closely related sequence, the Lucas numbers, $2, 1, 3, 4, 7, 11, 18, \dots$, obeys the identical recurrence and the identical growth rate, differing from the Fibonacci sequence only in its two starting values, a reminder that the recurrence itself, not the particular starting pair, is what fixes the growth rate to φ .

IX. CONCLUSION

The Fibonacci sequence reached Europe through a minor example in a book actually written to introduce mod-

ern digits, and it was already known, for an entirely unrelated reason, in Sanskrit poetic theory centuries earlier. It shows up genuinely, and specifically, in the spiral packing of seeds and leaves, through a real, provable property of the golden ratio's continued fraction, not just its numerical value. It does not show up, reliably, in the nautilus shell, despite being one of the most repeated claims about it. Its closed form can be reached two independent ways, once from the recurrence's own characteristic equation and once from a Taylor series in disguise, and both agree exactly. And it sits, unexpectedly, along the diagonals of a much older triangle of binomial coefficients.

Golden spiral built from squares of side 1, 1, 2, 3, 5, 8, 13, 21, 34

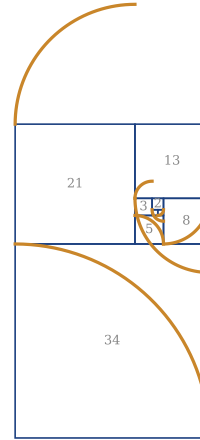


FIG. 5. The golden spiral, built from squares of side equal to consecutive Fibonacci numbers, tiling a rectangle with no gaps or overlaps. The spiral itself is the geometric signature of the sequence's exponential growth rate, φ per step.

None of this required assuming anything beyond what could be derived or checked directly.

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